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Third class problems of physics and technique that is intermediate between correct and incorrect classes of problems.

Summary of lectures courses.

Abstract

This paper presents a short summary of lecture courses that delivered at the department of the Applied Mathematics and Control Process of the Sanct-Petersburg State University since 1998 has been. This paper considers the third class of problems in mathematics, physics and technique which was recently discovered. This new class of problems is an addition calss to the previously known classes: correctly and incorrectly posed.

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Preface

This brochure is a short summary of lecture courses that the author delivers at the department of Applied Mathematics and Control Processes of the Saint-Petersburg State University. The necessity of such a course became quite evident after the existence of a new third class of problems in mathematics, physics and technique has been proved by St.-Petersburg state university scientists. This new class of problems is intermediate between already known other two classes of problems — correctly and incorrectly posed. The importance of this third class is evident and it is necessary to examine it in more details.

This brochure is only a short summary. A complete course of lectures has been prepared and will be published in the future. The author hopes that this brochure and other publications on the same question will allow to greatly improve teaching as at Petersburg state university and in other institutes.

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Introduction

Up to recent time all problems of mathematics, physics and technique were divided into two classes — correctly and incorrectly posed problems. In correctly posed problems to small changes (variations) of coefficients, parameters, initial or boundary conditions correspond small changes of solutions. In incorrectly posed problems during small variations of parameters, coefficients etc. large changes in solutions are possible.

This work is devoted to the investigation of a new third class of problems that has been recently found. This class of problems is so to say intermediate between correctly and incorrectly posed problems. Their correctness can change during equivalent transformations that are used in the course of their solution.

Firstly the investigation of this class of problems is interesting from theoretical point of view. And secondly it is interesting from practical point of view. Later we shall see it.

Since in practice small changes of coefficients, parameters etc. are usually inevitable for a long period time mathematicians worked at and studied correctly posed problems.

Only at the beginning of the 20th century an outstanding French mathematician J. Hadamar (1865–1963) paid attention to the existence of incorrectly posed problems and started their examination. J. Hadamar's first publications on this question is dated 1902 [1]. Later it turned out that many important problems of physics and technique (in particular, the problems of geophysics, geologic exploration by means of seismology methods, some problems of automatic control etc.) refer to incorrectly posed problems. And in the sixties (of the 20th century) academician A.N. Tikhonov and his school developed solution methods and methods of "regularization" of incorrectly posed problems. A. Tikhonov's works and of his school played an important role in the development of world mathematical science.

We shall not dwell on the methods of approach to the solution of incorrectly posed problems since they were described in detail in the works [1, 2, 3] and others. We must only pay attention to the fact that incorrectly posed problems require special methods of solution. Before finding the solution of any problem it is desirable to check its correctness since if we solve an incorrectly problem by usual methods (that are suitable for the solution of correctly posed problems) erroneous results will almost inevitably be obtained.

Therefore it was very important when in the period 1990 – 1997 a series of problems that can be referred to the third class of problems that were (so to say) intermediate between correctly and incorrectly posed problems was discovered. These new problems change their correctness during equivalent transformations of a mathematical model (including transformations used in the course of their solution).

An unexpected meeting with third class problems led to dangerous mistakes. And if we do not carefully examine this new class of problems the result will be the same.

Really, let an initial physical (or technical) problem that we must solve be incorrectly posed. Very often an initial mathematical model (that directly follows from laws of physics) of a given problem is not convenient for examining or it is cumbersome. In this case this mathematical model must be transformed to a more convenient or a normal form. Here we must surely use only equivalent transformations that do not change solutions.

Recollect that equivalent transformations are such transformations during which a set of solutions in an initial and transformed systems are identical [4]. But as we shall see later equivalent transformations can change correctness of the problem. Let the correctness of the problem change during transformation of a mathematical model to a "standard" (normal) form. Then during the examination of this model we shall admit that our problem is correctly posed. We shall solve it as a correct problem and thus we can obtain a quite erroneous result.

When we examined a well known problem of stability resources in automatic control systems it was just so. As a rule systems of differential equations of different orders are primary mathematical models of such systems that directly reflect processes occurring in elements of a system and connections between elements. Direct investigations of such systems is cumbersome. Therefore after quick operating technique has appeared a mathematical model consisting of different orders equations was reduced to a normal Cauchy form — to a system n equations of first order. This can be easily done by introducing new variables. The reduction to a normal Cauchy form later allowed us to apply a unique programs compilation, standard subprograms and it was very convenient. During the transfer to a normal Cauchy form surely only equivalent transformations were used. As a result all solutions of a transformed model coincide with solutions of an initial model.

Therefore stability computations in a transformed ("standard") model gave a correct result.

But gradually it was discovered that in the computation of stability resources errors that led to dangerous wrecks slipped in. And a real control system possessed very small stability resource but a computation by a mathematical model in a normal Cauchy form stated that stability resource is large and the system is reliable. Thus a compilation did not warn us about dangerous wrecks that are inevitable when stability resources are small (so much the more that during tests of a real machine it is not always possible to find out a value of stability resources.

In [5 – 10] this phenomenon was explained in the following way: equivalent (in a classical sense) transformations of equations were not obliged to leave unchanged such a system property as stability preservation during conventional methods of stability check and of stability resource check are not complete, they are not reliable and they can give erroneous answers in particular cases. And as a result they can become causes of dangerous wrecks. In the works [5 – 10] advanced methods of stability preservation computation that secure from wrecks were proposed. These advanced methods are based on the application of transformations that are equivalent not only in a classical sense but in a widened sense as well.

As warnings on the incompleteness of conventional methods during stability computation and on possible wrecks were serious (see [5 – 10]) animated discussions arose (it is partly reflected in [11, 12].

Up to now conclusions drawn in the works [5 – 10] are acknowledges by scientists. But the work on examination of transformations that are equivalent in a widened sense, i.e. that do not change correctness of the problem is far from completion.

Soon it was found out that correctness can change during equivalent (in a classical sense) transformations of not only differential but algebraic equations as well. In known problems of computing parameter values at which exist nonzero solutions of linear homogeneous equations system or computing such parameter values that turns to zero polynomial matrix determinant one of widely-spread methods is a method of successive transformations of an initial system into a system of less number of equations and of transferring a polynomial matrix into a matrix with less number of lines and columns. Surely during these transformations it is necessary to watch that all transformations be equivalent (in a classical sense) and correctness of an initial system is checked. Up to recent time no one noticed that transformations that are used can change correctness of a given problem. Then coefficients variations of an intermediate system that has been obtained in the course of calculations series of a system can lead to gross mistakes in the solution. They can be for example due to rounding off mistakes. This phenomenon was examined in [12].

Thus contours of a rather curious third class of problems in mathematics, physics and technique are drawn. This class of problems can change correctness during equivalent transformations. The investigation of this important class of problems is only at its start. Here great possibilities for interesting and effective scientific work are opened.

Up to now the examination of this third class of problems in mathematics, physics and technique can be in practice be of great importance because:

1. The possibility of preventing errors (wrecks and catastrophes that are connected with them) has been opened. These errors are due to incompleteness of conventional computation methods of stability resources in control systems that do not take into account possibilities for changing correctness.

2. Errors connected with the correctness change during transformations using in the course of its solution are prevented.

3. We can easily and simply solve (without regularization such problems whose incorrectness appeared only as a result of initial mathematic model transformations.

Later, I am sure, other practical application can be found.

1 Definitions.

We must distinguish rating values of coefficients and parameters in a mathematical model obtained as a result of an experiment or measurement (we shall denote them by a_i) and values that have been variated:

$$a_{i_v} = a_i(1 + \varepsilon_i) \tag{1}$$

where ε_i — numbers (positive or negative) that are small in comparison to 1. Numbers ε_i reflect inexactness of any experiment or measurement, an inevitable small drift of parameters in the course of time etc. We shall call values $\varepsilon_i a_i$ parameters variations. Evidently it is necessary to examine properties of a mathematic model of the system not only for rating values of parameters but it is necessary to take into account their variations that are inevitable in practice.

The definitions of variations (that we have adopted) according to (1) means that we shall investigate the influence of relative coefficients and parameters changes. If a rating value of some coefficient is equal to zero then their variations will be also zero. Zero does not variate. Cases when coefficient $a_i = 0$ are changed by a value that is not equal to zero (even infinitely small as is the case with singular-perturbing systems) we shall not satisfy our definition of a variation.

According to terminology of [1] we shall call a problem correct (or correctly posed) if its solution satisfies the following three conditions:

1. if it exists,

2. if it is the only one,

3. if to small changes of coefficients, parameters, initial or boundary conditions correspond to small changes of solutions (more accurate definitions of spaces in which the problem is examined etc. — see [1]).

Even if one of these conditions is not satisfied this problem is called incorrect (or incorrectly posed). We shall be interested only in correctness change that occurs if the third condition is not satisfied.

According to [4] we shall call equivalent in a classical sense such transformations during which a set of solutions in our problem of an initial and transformed system is identical.

Examples: the transfer of equations members from the left side to the right side with the change of the sigh; multiplication of all members by a number that is not equal to zero etc. The rules of equivalent transformations are well-known and they are already studied at secondary school. Let us introduce a new definition: we call transformations to be equivalent in a widened sense if:

1. they are equivalent in a classical sense,

2. if they do not change the correctness of our problem.

But not all transformations that are equivalent in a classical sense are also equivalent in a widened sense. Note that it is necessary to speak about equivalence or non-equivalence of the transformation only in relation to the problem under consideration. So if it is necessary to find values of a variable x that satisfy a system

$$\begin{cases} 3x + y = 4 \\ x + y = 2 \end{cases} \tag{2}$$

then if we subtract the second equation from the first one we shall come to equation $2x = 2$. It is equivalent in relation to the problem of a system (2). If it is necessary to find values x and y as well that satisfy to a system (2) then this same equation $2x = 2$ is already not equivalent to system (2).

Later we shall give examples and then the importance of a conception — equivalence in a widened sense that we have introduced will become clear. The rules and laws of this class of transformations are much more complex than the rules of transformations that are equivalent in a classical sense.

2 The most simple examples

Let us consider systems of linear homogeneous algebraic equations with parameter λ. The following system can serve as an example:

$$\begin{cases} \lambda x_1 - x_3 = 0 \\ x_1 - 2\lambda x_2 + x_3 = 0 \\ x_1 - x_2 = 0 \end{cases} \tag{3}$$

consisting of three equations with three variables. For such homogeneous systems always exist a trivial solution when all $x_i = 0$ but at some values of λ nonzero solutions can exist. Let us consider the problem of searching such values of parameter λ at which nonzero solutions exist. Such values of parameter λ are called principal values. And as it's well-known many practical problems are reduced to the funding of principal values.

It is well-known [13] that principal values coincide with roots coefficients matrix determinant for a linear homogeneous system — for system (3) it will be matrix

$$\begin{pmatrix} \lambda & 0 & -1 \\ 1 & -2\lambda & 1 \\ 1 & -1 & 0 \end{pmatrix} \tag{4}$$

The determinant of this matrix can be easily calculated. It is equal to $1 - \lambda$. Hence we can find the only one principal number $\lambda_1 = 1$. Computation problem by means of matrix determinant roots (4) is correct. We can easily convince ourselves in this if instead of matrix (4) we shall examine a matrix of coefficients that have been variated:

$$\begin{pmatrix} \lambda(1+\varepsilon_1) & 0 & -1(1+\varepsilon_2) \\ 1(1+\varepsilon_3) & -2\lambda(1+\varepsilon_4) & 1+\varepsilon_5 \\ 1+\varepsilon_6 & -1(1+\varepsilon_7) & 0 \end{pmatrix} \tag{5}$$

It is not difficult to check that if ε_i is small matrix (5) will have only one root that differ little from $\lambda_1 = 1$.

If it is necessary to find principal values in a large system of linear homogeneous equations with a parameter. Then a direct computation by large degree determinant is not convenient. As we know many practical problems lead to large systems with a large number of variables. Therefore usually a number of variables in our system is reduced and determinant degree is also reduced. Let us consider with what unexpected phenomena we encounter here.

In system (3) it is not difficult to exclude variable x_1: it is sufficient to multiply all members of the second equation from (3) by λ (note that $\lambda = 0$ is not a solution) and then the first equation must be subtracted from the second one. We shall obtain an equation that does not contain x_1.

The second equation that does not contain x_1 can be obtained by simple subtraction of the third equation from (3) from the second one. As a result of these equivalent transformations we shall came to the following system

$$\begin{cases} 2\lambda_2 x_2 - (\lambda + 1)x_3 = 0 \\ (1 - 2\lambda)x_2 + x_3 = 0 \end{cases} \tag{6}$$

System (6) has the same only one principal value $\lambda_1 = 1$ as system (3). This fact once more proves that both systems are equivalent (in a classical sense) to each other in relation to the problem of principal numbers computation. At the same time for system (6) this problem is already not correct. We can be easily convinced in this if we compute a determinant for system (6) with coefficients that have been variated:

$$\begin{vmatrix} 2(1+\varepsilon)\lambda^2 & -(\lambda+1) \\ 1 - 2\lambda & 1 \end{vmatrix} \tag{7}$$

(for simplicity let us variate only one coefficient from λ^2). Approximately, with the precision up to members having the highest degree of smallness we have $\lambda_1 = 1 - \varepsilon$; $\lambda_2 = \frac{2}{\varepsilon}$. Thus if ε is infinitely small system (6) has not one principal value but two. When $\varepsilon \to 0$ the second principal value does not at all tend to λ_1 and it disappears only if there is an exact equality $\varepsilon = 0$. Therefore during an equivalent transformation of system (3) into system (6) the correctness of the problem changed. And system (6) is equivalent to system (3) in a classical sense but it is not equivalent in a widened sense.

Many examples of such system in which during equivalent (in a classical sense) transformations correctness of a given problem changes can be seen in [10]. All problems of principal values computation for these systems refer to third class problem that are so to say intermediate between the classes of correctly posed problems.

Now let us examine to what consequences can an unexpected encounter with third class problems lead. If all coefficients are integral numbers (as in system (3)) then no annoyances occur. But if coefficients are fractional numbers then even inevitable rounding off errors can lead to the following. Coefficients variations in a transformed system can become independent of coefficients variations in an initial system that we have checked in correctness. And this can lead to gross mistakes in computations. Therefore a new errors source appears in computations that is different from error sources that have been known before. By all means this error source can be overcome. But for this it is necessary to know about the existence of a third class problems in physics and technique that are different of two other known classes. It is necessary to know as much as the most simple rules of treating problems of this class.

In monograph [14] this question was examined in more details in supplementary chapters of the second edition). This book can be used in teaching together with, in addition to this short summary of lectures. As we have already indicated a complete course of lectures is prepared for publication. But in 1998 the editors underwent financial difficulties and the publication of the course was detained.

In the next chapter we shall examine examples referring to a special case of correctness change — when stability resource change during equivalent transformations of simple differential equations systems, i.e. — mathematical models of electric machines, control systems etc. Just this specific case recently provoked the most acute discussions and considerations.

3 Applications to electric machines, control systems, to wrecks prevention. Correctness changes in differential equations systems.

Mathematical models in which correctness changes during equivalent transformations were previously found during the examination of wrecks in automatic control systems. The analysis of wrecks causes that were connected with correctness changes (and in particular with the change of stability resource during transformations of equations in variations of parameters) allowed to prevent one of wrecks cause.

In control systems we encounter correctness change in mathematical models that are systems of simple differential equations. Let us consider the following simple example: a system of regulating rotation frequency in electric drives with constant current machines. It can be written in the form:

$$\begin{cases} \dot{x}_1 = -2x_1 + x_2 + u \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = -x_2 - 2x_3 \end{cases} \tag{8}$$

$$u = -x_1 - 2x_2 - x_3 \tag{9}$$

where x_1 — deviation of rotation frequency from a given value, x_2 and x_3 — perturbing interaction, u — controlling interaction, armature current. Equation (8) are control object equation (of constant current electrical machine), equation (9) are equation of a regulator, feedback chains. Equation (8) and (9) are written for rating values of parameters which have been rounded of up to integral numbers for checking convenience of further computations (see the example). If we wish to analyse the influence of some parameters (for example — a mechanical time constant T) on transient processes in a control system then equation (8) can be written in a more general form with the following parameter:

$$\begin{cases} \dot{x}_1 = -\frac{2}{T}x_1 + \frac{1}{T}x_2 + \frac{1}{T}u \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = -x_2 - 2x_3 \end{cases} \tag{10}$$

Equation (8) is a special case of equation (10). It is written for a rating value of parameter T for $T = 1$. A system of equations (8) and (9) (if we consider them jointly) describe processes that occur in a closed control system. While solving this system first of all we must find its characteristic polynom:

$$\lambda^3 + 5\lambda^2 + 7\lambda + 3 = (\lambda + 1)^2 \cdot (\lambda + 3) \tag{11}$$

whose all roots ($\lambda_1 = -3$; $\lambda_2 = \lambda_3 = 1$) lie in the left semi-plane of a complex variable. They lie not far from an imaginary axis. If we know the roots it is not difficult to write a general solutions of a system (8)–(9) for all variables. So for x_1 we have

$$x_1 = c_1 e^{-3t} + (c_2 t + c_3) e^{-t} \tag{12}$$

all solutions of system (8)–(9) are asymptotically stable. Now let us check stability resource of system (10) in possible variations of parameter T. At the same time it will be the check of solution in this parameter. For this it is necessary to find a characteristic polynom for a system of equations (9)–(10). This polynom is of the form:

$$(T\lambda + 3)(\lambda + 1)^2 \tag{13}$$

with roots $\lambda_1 = -\frac{3}{T}$; $\lambda_2 = \lambda_3 = -1$ that lie in the left semi-plane of a complex variable for all T in the interval $0 < T < \infty$. Therefore system (9)–(10) preserves stability not only during small deviations of parameter T from a rating value $T = 1$ but during great deviations. The system has a large stability resource in parameter T. By means of similar checks it can be confirmed that there are good resources of stability in variations of any coefficients in system (8)–(9). System (8)–(9) is correct.

Now let us suppose that (as it often occurs) in a feedback channel only variable x_1 can be directly used. But we are eager to preserve the same transient processes that take place in a control system according to formula (12). For this it is sufficient to exclude variables x_2 and x_3 from equations (8) and (9). Naturally it is necessary to use here only equivalent (in a classical sense) transformations. After the exclusion of x_2 and x_3 we obtain the following system:

$$(D^3 + 4D^2 + 5D + 2)x_1 = (D^2 + 2D + 1)u \tag{14}$$

$$(D^2 + 4D + 5)x_1 = (D + 1)u \tag{13}$$

where $D = \frac{d}{dt}$ — differentiation operator. Now only variable x_1 is used in a feedback channel.

It is not difficult to check that a characteristic polynom from system (14)–(15) is of the form (11) as before. But a general solution has the form (12). This fact confirms once more that system (14)–(15) is able to loose stability is some coefficients deviate from rating values infinitely little (in contrast to system (8)–(9)). For example, let coefficient in D^2u from equation (14) be equal to some number a but not to 1. It is not difficult to check that if $a > 1$ (or even if a is more than 1 in a small number ε) a characteristic polynom from system (14)–(15) will have a positive root and stability when $a > 1$ will be lost. System (14)–(15) has a zero stability resource in variations of some coefficients and it is an incorrect system (to be more exact — the problem of finding stability for system (14)–(15) is incorrect). At the same time system (14)–(15) is equivalent (in a classical sense but not in a widened sense) to system (8)–(9). Now let us analyse stability resource in a certain parameter T — mechanical time constant for electric motor. Therefore it is necessary to exclude variables x_1 and x_2 from equation (10). Jousted of equation (14) we shall obtain

$$(TD^3 + (2+2T)D^2 + (4+T)D + 2)x_1 = (D^2 + 2D + 1)u \tag{16}$$

(if it naturally coincides with equation (14)) but equation (15) will remain unchanged since a regulator and its all parameters do not naturally depend on parameters variations of an electric motor. While computing a characteristic polynom from system (15)–(16) we can conclude that it is of the form:

$$[(1-T)\lambda^2 + (2-T)\lambda + 3](\lambda+1)^2 \tag{17}$$

but when $T > 1$ it has a positive root. Therefore if parameter T will exceed a rating value $T = 1$ by even infinitely small number then stability of a closed system will be lost.

Note that system (14)–(15) can be reduced to a normal Cauchy form by introducing new variables

$$\begin{cases} x_2 = Dx_1 - u \\ x_3 = Dx_2 \end{cases} \tag{18}$$

(it is an equivalent transformation). But the problem of checking stability resource here changes. For systems (8)–(9) and (14)–(15) the problem of checking stability is an example of third class problems. This class of problems changes its correctness during equivalent transformations used in the course of their solution. In works [5–10] many such problems are considered.

Now let us examine consequences that follow from these examples. Strictly speaking all these consequences already follow from the existence of one example with systems (8)–(9) and (14)–(15). All other examples only stress that systems of the type (8)–(9) and (14)–(15) are not rare at all.

Consequence 1. Conventional methods of checking stability and its resource that are used at all constructional-projection bureaus in Russia and abroad are apriori not complete. They are based on the examination of a characteristic polynom or matrix of a coefficients from a normal Cauchy form. And not in all cases they can give a correct answers. The same can be said about methods that apply classical amplitude versus frequency response characteristics and about the newest methods that are using Kharitonov's theorem [15] and a theory of robust stability [16]. As it was indicated in [8] no investigation of a characteristic polynom or matrix of coefficients in a normal Cauchy form can always give a satisfactory answer to the question of stability re sources. Errors arise in such cases where we encounter such problems that refer to the third class. In order to obtain trustworthy answers conventional methods must be completed by computations described in [5–10].

Consequence 2. Methods based on the second Lyapunov method and the creation of Lyapunov functions does not also secure trustworthy answers. A system that we are examining can have a good Lyapunov function and at the same time it can have a zero stability resource. Such examples do not refute the correctness of Lyapunov theorem. But a stable system and at the same time it has a zero stability resource from a practical point of view is not at all better than unstable system. It is even more dangerous.

Questions connected with Lyapunov functions were considered in more details in the work [10]. This work can be used in teaching. Note that a combination of Lyapunov function existence with a zero stability resource that depreciates it arises just when we encounter third class problems.

Consequence 3. Since it is difficult to find computation errors for modern complex control systems during tests these errors can become the cause of dangerous wrecks and even catastrophes.

Really if a system is not stable then we shall find it out at once during tests. If a system is stable but if it is able to loose stability at parameters variations of a certain sign or at some combinations of variations signs then it is often impossible to find out this dangerous variations combinations. In a complex control system that, for example, includes forty parameters a number of possible combinations of their signs is equal to 2^{40} and it is more than 10^{12}. It is impossible to check them. Therefore the following situation is possible: when due to tests a control system is acknowledged to be stable, it is mounted on an important object but later during exploitation small drift of parameters occurs. Then this same system can loose stability at any arbitrary unexpected moment and a wrecking situation or even a direct wreck arises. In [10] are given that some of well-known wrecks and catastrophes of recent years have arisen due to just this cause.

In order to prevent wrecks it is necessary to apply additional computations and checks of stability resources. Theoretical bases of such checks are described in [5–10]. In order to use them in practice it is necessary to develop programs compilations that is suitable for the production that is manufactured by an enterprise or a firm. In recent years almost all enterprises in Russia are under acute financial difficulties. Therefore they are not ready for ordering and financing new sets of programs compilations. Practical application of advanced computational methods is greatly detained. Surely a firm that is applying these advanced computation methods at their enterprises gain advantages over their competitors. Their production will be more reliable, there would be less probability of wrecks or refusals of the equipment.

To our regret we can sooner set examples of negative experience. So in the period 1994–98 a substantial part of auxiliary equipment (pumps, electric drives, controllers etc.) have been substituted by new ones at the Leningrad atomic electric station (LAES) situated near St.-Petersburg. St.-Petersburg state university proposed to carry out additional checking of stability resources of this new equipment. LAES itself or together with the Administration of the St.-Petersburg Governor had to pay scientists of Petersburg state university for the programs compilation. It casts 20 thousand dollars (it's a ruble equivalent of that time). Neither in 1994 nor in 1995 money was found. Nothing was done. Ecologists started worrying since an increased probability of wrecks or refusals at LAES became evident. In Petersburg newspapers articles criticizing the Administration for inattention to the security of city's population appeared. The Governor did not react to the warnings of University scientists.

Meanwhile in October 1996 St.-Petersburg and its neighbor Stockholm put forward their applications to the International Olympic Committee for carrying out Olympic Games 2004 in their cities. The Swedes knew the situation at LAES very well. They were greatly surprised that the Petersburg

Governor paid no attention to the warnings of its own university. As a result on the 7th of March, 1997 the International Olympic Committee declined the application of St.-Petersburg even on the first round on the grounds that St.-Petersburg is a dangerous city. But Stockholm continued to take part in the competition and only on the second round the preference was paid to Athens. And St.-Petersburg's budget lost 129 billions rubles (an equivalent of 20 million dollars at that time) — principally on advertisements.

Thus up to now the developments of university scientists on the computation of stability resources have not lead to the increase of safety of the city. But gradually such developments find different applications.

In next chapter we shall dwell on the following question: how the notion of a class of problems that change its correctness and the notion of transformations that are equivalent not only in a classical but also in a widened sense gradually developed.

4 The conception development of transformations that are equivalent in a widened sense.

In the course of optimal control theory development a conception of transformations that are equivalent in a widened sense became necessary. It was found that systems, with the help of which quality criterion achieves a maximum, often possess a zero stability resource in parameters variations. This phenomenon was evident even on the first steps of development of the optimization theory by means of squared quality criterion. Therefore attempts in using optimal systems in practice led to wrecks. And as a result trust in optimal control theory was harmed and the possibility of its practical application was closed. For a long period of time searches for such optimization methods that would be free of this drawback continued. More and more algorithms we proposed (see [17–20]) but stability loss was found in them as well. For the last time in 1973 [2] P.V. Nadezhdin found a zero stability resource in systems that were synthesized according to the next algorithm shortly before this algorithm was proposed by V.B. Larionov, K.I. Naumenko and V.N. Suntzev. And Nadezhdin considered it a drawback in the algorithm. But in 1973 in a monograph [22] it was shown that the minimum of quality criterion objectively often lies on the boundary of parameters stability. Therefore searches for an algorithm that will always combine the quality criterion minimum with nonzero stability resource have no sense.

After a monograph [22] was published the development of optimal control theory took another course. Vain searches of an algorithm that secures the combination of parameter stability resource with an exact minimum of quality criterion were stopped. Instead stability resources have to be provided for and this is a special requirement for whose realization (in such cases when it was not realized automatically) it is necessary to pay by some increase of quality criterion. Methods that allow to synthesize optimal systems with good stability resources were developed. The least losses in quality criterion secures the following method developed in [23]: analytical approximation of perturbing interactions spectrum in a range of frequencies that are the least essential for a given system is corrected. In the main this method secured the regularization of an incorrectly posed problem of optimization, the transformation of incorrectly posed problem into a correctly posed problem although in monograph [23, 24] this terminology was not used.

During the synthesis of many-dimensional optimal systems (many examples can be found in [23, 24, 5] we often encounter such cases that often seem to be paradoxical: when equivalent transformations change stability resources. And then a system that is stable in parameters turn into a system that is unstable in parameters and vice verse. The analysis of these examples allowed to draw the following conclusion ([5], p. 230): such phenomena are quite lawful since stability preservation during parameters variations (contrary to stability as such) is not a property of a system itself but it is a property of its surrounding, its neighborhood in parameters space. Therefore simple equivalent transformations must not preserve this property.

But from this statement (see [5]) it directly follows that conventional methods of computing stability and its resource that are based on the analysis of a normal Cauchy form are not complete and they can not guarantee correct computation results. This is true not only for a special case of optimal control systems that were considered in [5] but for any system. Since this conclusion published in [6, 7] raised a very important question on the necessity of revising (or to be more exact of supplementing) computation methods that are used in Russia and in other countries during projection or construction of different equipment it provoked a stormy and acute discussions. So in St.-Petersburg and Moscow such discussions took place at scientific seminars and on pages of scientific magazines. In the course of these discussions a lot of deep and sharp objections were expressed. The analysis of these objects was published in [10], pp 36–46. Finally after acute discussions published in [8, 9] and responses to them [11, 12] main results of works [8, 9] were acknowledged by scientists.

Let us also note that results published in [5, 8 – 10] can be considered as a continuation and further development V.I. Zubov, V.I. Vorotnikov, V.S. Yermolin and others [25, 26] that were dedicated to the examination of stability in a part of variables. In the course of these examinations the phenomenon of parameters stability change during equivalent transformations of equations used for obtaining μ — system was repeatedly observed. But finally that set of phenomena that could not be explained earlier was explained when conception of transformations equivalent in widened sense was introduced. It became clear that the change of parameters stability during equivalent transformations as for optimal systems so for systems that are stable in a part of variables occurred when the applied transformations were equivalent in a classical sense but not in a widened sense.

As to the methods of discovering stability small resources of their increase and wrecking prevention they have been developed when optimal systems were investigated (see [5], pp. 220 – 230). There it is shown now systems that loose stability during parameters small change (unstable systems) can be turned into stable ones. Surely methods described in [5] can be applied not only to optimal but to any systems. It is well-known that any systems can be considered as optimal according to some quality criterion. But at the same time the application of a methods stated in [5] required good programs compilation.

Thus the first examples of mathematical models that change some its important properties (and in particular — parameters stability) after equivalent transformations were discovered among optimal control systems. A little later similar phenomena were discovered in systems of algebraic equations [10].

The following conclusion was drawn: such examples can be united into one (third) class of problems in mathematics, physics and technique. This class of problems is intermediate between correctly and incorrectly posed problems. And transformations that are equivalent in a widened sense should defined as transformations that preserve correctness of problem that is solved. Stability loss during infinitely small parameters variations is a special case of incorrectness.

The examination of the third class problems of mathematics, physics and technique (as well as the properties examination of transformations that are equivalent in a widened sense) has only started. Surely a lot of problems that refer to the third class will be discovered in near future. And lows that they follow will be as well opened. Here much interesting and fruitful scientific investigation is before us.

But examinations that have already been accomplished allow us to discover causes of many contradictions and paradoxes in a conventional control theory. For example the following well-known prohibition: "it is not allowed to reduced by non-Hurvitz operators polynom that is in the left and right sides of a differential equation" is illogical since such a reduction does not violate equivalence. Besides, during operational calculations this reduction is widely used. No more logical are other numerous and contradictory to each other prohibitions on some transformations in different sections of automatic control theory. Besides it remained not clear whether all important prohibitions were taken into account and whether some prohibited transformations that have not as yet been found become the source of errors. After the theory of transformations equivalent in a widened sense has been developed all just "prohibitions" that were put forward by practice found the following simple and unique explanation. If we are interested in solutions of equations systems as such then any equivalent (in a classical sense) transformations of a system are admissible. If we are interested in solutions behavior during inevitable parameters variations (in practice) then only a more close class of transformations that are equivalent in a widened sense are admissible. Hence — "prohibitions". In fact these "prohibitions" that are well-known in automatic control theory deal with transformations that are equivalent in a classical sense but not in a widened sense.

5 General principles of defining problems of the third class.

When problems referring to one or the other class are defined (to a class of correct, incorrect or intermediate between the two other classes) it is necessary to take into account that this definition depends on the following factors:

1. a type of a mathematic model that we examine
2. a solved problem
3. a type of transformation that is used during its solution.

Let us consider a known problem of searching parameter λ principal values, i.e. values at which nonzero solutions of linear homogeneous equations system in some of which (but certainly — not in all!) parameter λ enters exist. As it is known many important practical problems are reduced to the solution of this problem. For example a problem of checking stability and damping velocity of transient processes in control systems, the computation off small oscillations frequency and their damping in mechanical and electrical systems etc.

A mathematical model to which many of these problems are reduced can be written in the form of n equations system:

$$(A - \lambda \bar{E})x = 0 \tag{19}$$

where x — n-dimensional vector of variables x_i, A — a square matrix of coefficients ($n \times n$ dimension), $\bar{E}$ — quasi-unit matrix in which there are $n - r$ units and r zeros on a main diagonal and all other elements are zeros. Note that in equations (19) just a quasi-unit matrix but not a unit matrix exists since its presence reflects such properties of practical problems as the existence of holonomic relations between variables when frequencies of small oscillations in mechanical systems are sought, the presence of feedbacks without differentiation operators in control systems etc.

Therefore our problem is a generalization of a known computation problem of matrixes principal values [13]. It turns into the last one if $r = 0$. Let us suppose that we are solving this problem by successful excluding variables x_1, x_2 etc. until we come to a system of two equations:

$$\begin{cases} A_1(\lambda)x_{n-1} + A_2(\lambda)x_n = 0 \\ A_3(\lambda)x_{n-1} + A_4(\lambda)x_n = 0 \end{cases} \tag{20}$$

where $A_1, \ldots, A_4$ — parameter λ polynomials. A determinant of system (20) is of the form:

$$\begin{vmatrix} A_1(\lambda) & A_2(\lambda) \\ A_3(\lambda) & A_4(\lambda) \end{vmatrix} \tag{21}$$

and principal values of system (19) are among its roots.

Now let us examine correctness changes during the transfer from system (19) to system (20) that takes place by means of successive multiplications and additions. We multiply the first pair of equations from a system (19) by such numbers (or parameter λ polynomials) that coefficients in variable x_1 be equal in a modules and be contrary by a sign and then we add them together. If we carry out the same operation with the second and the third equations from system (19) then — with the third and the fourth equations etc. we exclude variable x_1 and come to a system of $n-1$ equations. If we repeat the same procedure of multiplications and additions we shall come to system (20).

In a complete text of lectures the following theorems on the correctness change are proved:

Theorem 5.1 If $r = 0$ then during the transfer from (19) to (20) no correctness change of the problem of computing principal values does not occur.

Theorem 5.2 If $r = 1$ and $n \geq 4$ then when $a_{n-1;n}a_{nn} = 0$ correctness changes. If $a_{n-1;n}a_{nn} \neq 0$ then correctness is preserved.

From theorems 5.1 and 5.2 simple consequences follow:

1. correctness changes occur in principal values generalized problem if there are holonomic equations that connect variables. In a classical problem of principal values [13] no correctness change occur. Just due this phenomenon correctness change was found so late.

2. During formal variables exclusion carried out on a digital computer by the order of their indexes mistakes in computations are possible. They are due to correctness change. But mistakes can be easily avoided if variables exclusion is started from equations that do not contain λ.

Note that variables exclusion by means of multiplications and additions often lead to the appearance of redundant roots in determinant (21). If we have come to mathematical model (19) from a problem of control systems synthesis with good transient processes then the appearance of redundant roots means that after variables exclusion transient processes also change. And this is undesirable. Therefore an another method of variables exclusion from a system of a type (19) that was in details described in [5], pp. 216–230 was proposed. It includes several successive multiplications by λ and then a matrix equation that was obtained was solved. By using this exclusion method if, for example, $n = 4$ and $r = 1$ correctness change occurs even for any values of matrix A coefficients in (19) to (20). But it occurs only if $a_{31}a_{44} = 0$ when we turn to variables exclusion by multiplications and additions.

Thus for the same problem of defining principal values, for the same equations system one exclusion method leads to correctness loss but another method — does not.

In works [27] and [28] a simple method of "degrees matrix" was proposed. It allows in rather a simple way to find necessary conditions for possible correctness change if system dimension is decreased not only for systems of the type (19) but even for more complex systems where instead of any coefficients a_{ij} there is an arbitrary degree.

"Degrees matrix" method can be described in the following way. Each of polynomial coefficients in system (19) or in any similar system is substituted by an integral number equal polynom degree. After this we carry out such transformations of each pair of lines in "degrees matrix" (that we have obtained) that occur in this lines pair during the exclusion of the next variable. First of all numbers of the first line are increased by a number situated in the extreme left cell of the second line and all numbers of the second line are increased by a number situated in the extreme left cell of a first line. After this the obtained lines pair is substituted by one whose each element is equal to the maximal one from corresponding elements of the first and the second lines and an extreme left element is thrown away. After we have carried out these simple transformations with each pair of lines — the first, the second ones then with the second and third one etc. we obtain a new "degrees matrix" that is of less dimensions. By repeating these transformations we come to matrix of dimension 2×2. If in this matrix sums on diagonals are equal this means that for an obtained system (after excluding $n-2$ variables) the problem of principal values computation can be incorrect. Similarly necessary conditions for possible change of correctness and if $n-3$ are excluded. A detailed description of "degrees matrix" methods is given in [27].

Example: for system (3) degrees matrix is of the form:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{22}$$

and after variable x_1 exclusion it turns into a matrix:

$$\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \tag{23}$$

in which sums on diagonals are equal. This means that after x_1 exclusion by means of equivalent transformations the problem of principal values computation can become incorrect. In section 2 we directly excluded variable x_1 from system (3) and could make a conclusion that it was just so.

For a classical problem of computing matrix $(A - \lambda E)$ principal values then degrees matrix (when $n = 4$) is of the form:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{24}$$

If we transform it according to the above rules we obtain a simple chain:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix} \tag{25}$$

at the end of which is a degrees matrix in which sums on diagonals are not equal. Once more this fact allows us to conclude that in classical problem of computing matrixes principal values there is no less of correctness.

If we last equation from equations system does not contain λ (i.e. $n = 4$, $r = 1$) then the last line in matrix (24) is substituted by a line $(0, 0, 0, 0)$ and we come to the following chain of transformations:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} \tag{26}$$

We shall obtain a matrix having equal sums on diagonals. This means that if $n = 4$ and $r = 1$ correctness loss is possible when two variables are excluded.

When $n = 5$ and $r = 2$ correctness loss is also possible since a corresponding chain of "degrees matrixes" is of the form:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 3 & 2 & 2 \\ 1 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 5 & 4 \\ 3 & 2 \end{pmatrix} \qquad (27)$$

Exactly in the same way it is also easy to formulate a theorem on the possibility of correctness change for any n and any r.

The problem of correctness change during transformations of differential equations systems and mathematical models of control systems can be easily solved on the basis of "degrees matrix" methods. It is sufficient to change differentiation operator by a parameter λ. So for example the check of correctness change possibility in system (8)–(9) that is considered in section 3 is reduces to the construction of a chain (26).

At the same time it is necessary to again stress that not all properties of the most interesting third class of problems in mathematics, physics and technique. There is a wide field for interesting and fruitful scientific developments.

6 Methodical explanations

This short summary naturally can not replace a complete course of lectures but it can be used together with other publications [5–10, 14, 27, 28] by the teacher. Additional examples of such control systems that change correctness during transformations are given in [6–10]. The methods of regularization of incorrect control problems (the transformation of "unstable" systems into "stable" ones) is given in [5]. In [5] methods of checking possible unstableness and methods of defining systems that have small stability resources are also given. Detailed analysis of transformations that are equivalent in a classical and in a widened sense is given in [10]. There are also analyzed discussions and objections that arose after the publications [5, 6, 7]. Examples of algebraic equations systems for which computation problem of principal values change correctness as a result of equivalent transformations are given in [10, 27, 28] and in supplementary chapters of a monograph [14]. Method of "degrees matrixes" is described in [27]. Works [27], [28] can be found only in an electronic variant, write (E-mail): "Dmitry.Frolenkov@pobox.spbu.ru" or "denissov@apmath.spbu.ru". Additional materials on a course of lectures dedicated to third class problems in physics and technique that were carried out at St.-Petersburg state university can be sent. Besides methods of additional computations that guarantee from wrecks and catastrophes can be sent to you as well.

References

1. Tikhonov A.N., Arssenin V.Ya. Solution methods of incorrectly posed problems. M., Nauka (Science), 1978

2. Ivanov V.K., Vassin V.V., Tanava V.P. Theory of linear incorrectly posed problems and its applications. M., Nauka, 1978.

3. Tikhonov A.N., Goncharsky A.V., Stepanov V.V, Yagola A.G. Numerical solution methods of incorrectly posed problems. M., Nauka, 1990, 230 pp.

4.Mathematical encyclopedia, v. 4, 1977, p. 800

5. Petrov Yu.P. Synthesis of optimal control systems when perturbing forces are known incompletely. Izdatelstvo Leningradskogo Universiteta (Leningrad State University edition), 1987, 289 pp.

6. Petrov Yu.P. On hidden dangers in conventional stability check methods. Izvestiya VUZ. Electromekanika (High school informations. Electromechanics), 1991, N 11, p. 106–108

7. Petrov Yu.P. Computation of control systems that preserve stability during parameters variations. St.-Petersburg, 1992, 35 pp.

8. Petrov Yu.P. Stability of linear systems at parameters variations. Avtomatika i telemekanika (Automatics and telemechanics), 1994, N 11, p. 186–189

9. Petrov Yu.P. Wrecks prevention in control systems. Izvestiya VUZ. Electromekanika (High school informations. Electromechanics), 1994, N 1–2, p. 37–40

10. Petrov Yu.P. Mathematical model and physical reality. St.-Petersburg, 58 pp.

11. Gaiduk A.R. On the investigation of linear systems stability. Avtomatika i telemekanika (Automatics and telemechanics), 1997, N 3, p. 153–160

12. Gaiduk A.R. Control systems synthesis when there are weak conditions for the wholeness of objects. Avtomatika i telemekanika (Automatics and telemechanics), 1997, N 4, p. 133–144

13. Wilkinson J.H. Algebraic problem of principal values. M., Nauka, 1970, 564 pp.

14. Petrov Yu.P. Chervyakov V.V. Stabilization system of boring vessels. Second edition. Izdatelstvo Leningradskogo Universiteta (Leningrad State University edition), 1997, 261 pp.

15. Kharitonov V.L. On asymptotic stability of equilibrium state in a family of linear differential equations. Differentzialnyie uravneniya (Differential equations), 1978, N 11

16. Polyak B.T., Tzipkin Ya.Z. Frequency criterion of robust stability and aperiodicity of linear systems. Avtomatika i telemekanika (Automatics and telemechanics), 1990, N 9

17. Chang Sh. Optimal automatic control systems synthesis. M., Mashinostroenie (Machine building), 1964, 440 pp.

18. Marriem C. Optimization theory and control systems with feedback computation. M., Mir (World), 1967, 549 pp.

19. Letov A.M. Flight dynamics and control. M., Nauka, 1969, 360 pp.

20. Larrin V.B. Naumenko K.I., Suntzev V.N. Synthesis of optimal feedback linear systems. Kiev, Naukova Dumka (Scientific thoughts), 1973, 150 pp.

21. Nadezdin P.V. On robustness loss during simple transformations of differential equations in control systems. Avtomatika i telemekanika (Automatics and telemechanics), 1973, N 1, p. 185–187

22. Petrov Yu.P. Optimization of controlled systems subjected to the influence of wind and rough sea. L., Sudostroenie (Shipbuilding), 1973, 216 pp.

23. Petrov Yu.P. Variational methods in optimal control theory (second edition). L., Energiya (Energetics), 1977, 280 pp.

24. Abdullayev N.D., Petrov Yu.P. Theory and projection methods of optimal regulators. L., Energoatomizdat (Energy-atomic publications), 1985, 240 pp.

25. Zubov V.I. Mathematic methods of automatic control systems examination. L., Mashinostroenie (Machine building), 1974, 335 pp.

26. Vorotnokov V.I. Stability of dynamic systems in relation to a part of variables. M., Nauka, 1991, 284 pp.

27. Petrov Yu.P., Frolenkov D.B. Correctness change during equations transformation. E-mail: Dmitry.Frolenkov@pobox.spbu.ru

28. Petrov Yu.P., Frolenkov D.B. Variants of correctness change for a class of problems that are intermediate between correctly and incorrectly posed problems. E-mail: Dmitry.Frolenkov@pobox.spbu.ru